A Strong-Scatter Theory of Ionospheric Scintillations for Two-Component Power Law Irregularity Spectra

Charles S. Carrano, Charles L. Rino
Institute for Scientific Research
Boston College

ABSTRACT

We extend the phase screen power law theory for ionospheric scintillation to account for the case where the refractive index irregularities follow a two-component power law spectrum. The two-component model includes, as special cases, an unmodified power law and a modified power law with an outer scale or inner scale. As such, it provides a useful framework for investigating the effects of a spectral break on the scintillation statistics. Using this spectral model, we solve the 4th moment equation governing the intensity fluctuations for the case of two-dimensional field-aligned ionospheric irregularities. A specific normalization is invoked to exploit the self-similar properties of the problem and achieve a universal scaling, such that different combinations of perturbation strength, propagation distance, and frequency produce exactly the same results. The numerical algorithm is novel in that it employs specialized quadrature algorithms and a Python library for arbitrary-precision floating-point arithmetic (mpmath). These advancements enable simulation of significantly stronger scattering conditions than previously possible, enabling us to validate new theoretical predictions for the behavior of the scintillation index and intensity correlation length under strong scatter conditions.

1. STRONG SCATTER THEORY

We begin by assuming the effects of the disturbed ionosphere with statistically homogeneous refractive index fluctuations can be adequately represented by a thin phase changing screen characterized by the phase spectral density function $\Phi_{\delta\phi}(q)$. We further assume the fluctuations are formally two-dimensional, i.e. infinitely elongated field-aligned structure typical of the disturbed low-latitude ionosphere. The corresponding phase structure function is

$$D_{\delta\phi}(r) = \int_{-\infty}^{\infty} 2\left[1-\cos(qr)\right]\Phi_{\delta\phi}(q)\,dq/(2\pi) \tag{1}$$

where r is the spatial separation and q is the angular wavenumber in the direction of phase variation in the screen. Under the paraxial approximation the equation governing the 4th moment of intensity fluctuations assumes parabolic form [Ishimaru, 1997; Yeh and Liu, 1982]. For the case of a normally incident plane wave, the solution for the intensity spectral density function (SDF) following traversal of the screen can be expressed as [Gochelashvily and Shishov, 1971; Rino, 1979]:

$$\Phi_I(q) = \int_{-\infty}^{\infty} \exp\left[-\gamma\left(r, q\rho_F^2\right)\right]\exp\left(-iqr\right)dr \tag{2}$$

where $\rho_F = \sqrt{z/k}$ is the Fresnel scale, z is the propagation distance past the screen and k is the free-space wavenumber of the radio wave. The structure interaction term that appears in (2) is given by

$$\gamma\left(r_1,r_2\right) = D_{\delta\phi}\left(r_1\right) + D_{\delta\phi}\left(r_2\right) - \tfrac{1}{2}D_{\delta\phi}\left(r_1+r_2\right) - \tfrac{1}{2}D_{\delta\phi}\left(r_1-r_2\right) \tag{3}$$

Using the half-angle angle identity, the structure interaction function can be written equivalently as [Rino 1979, equation 4]:

$$\gamma(r_1,r_2)=8\int_{-\infty}^{\infty}\Phi_{\delta\phi}(q)\sin^2(r_1 q/2)\sin^2(r_2 q/2)\frac{dq}{2\pi} \tag{4}$$

The scintillation index, S_4, is obtained by integrating the intensity SDF over all wavenumbers:

$$S_4^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_I(q)\,dq-1 \tag{5}$$

The unity term in (5) effectively removes the mean wave intensity. The intensity correlation function is the inverse Fourier transform of the intensity spectrum:

$$R(r)=\int_{-\infty}^{\infty}\Phi_I(q)\cos(qr)\frac{dq}{2\pi}, \tag{6}$$

which is commonly presented in normalized form as

$$\rho(r)=\left[R(r)-1\right]/S_4^2. \tag{7}$$

The correlation length r_c is defined as the spatial separation for which the intensity correlation decreases to 50%, i.e. $\rho(r_c) = 1/2$.

Two-Component Structure Model

Rino and Carrano [2013] proposed the following 2-component model for the phase SDF corresponding to a 2D irregularity model:

$$\Phi_{\delta\phi}(q)=C_p'\begin{cases}|q|^{-p_1}, & q\le q_0\\ q_0^{p_2-p_1}|q|^{-p_2}, & q>q_0\end{cases} \tag{8}$$

The parameter C_p' is the strength of the phase SDF evaluated at 1 rad/m, and the prime is used to distinguish this from the phase spectral strength for a 3D irregularity model. The model (8) has a spectral break at the wavenumber q_0. The break wavenumber appears in the lower branch to force continuity of the spectrum at q_0. With this form p_1=p_2 gives an unmodified power law, and p_1=0 gives an outer scale. In addition, the case p_1<3, p_2>3 can be made to resemble an inner scale. Henceforth, we consider only cases of practical interest with $1<p_1\le p_2<5$.

Self-Similar Form and Universal Scaling

The Fresnel scale arises naturally in diffraction calculations, even in a power-law environment which lacks a dominant scale. We use the Fresnel scale to define a normalized spatial separation variable $\eta=r/\rho_F$ and a normalized wavenumber $\mu=q\rho_F$. This approach differs from that taken by previous authors who normalize by the outer scale [Uscinsci et al., 1981; Bhattacharyya et al., 1992]. Since the outer scale may or may not have a strong influence on the intensity field, depending on the spectral shape and strength of scatter, we argue the outer scale is not ideal for normalization. In a power-law environment the Fresnel scale is more fundamental and it preserves the self-similar scaling of the problem.

In terms of the normalized variables, the phase SDF can be written

$$\mathrm{P}(\mu)=\begin{cases}U_1\mu^{-p_1}, & \mu\le\mu_0\\ U_2\mu^{-p_2}, & \mu>\mu_0\end{cases} \tag{9}$$

where the scattering strength parameters are given by

$$U_1 = C_p' \, \rho_F^{p_1-1}, \quad U_2 = C_p' \, q_0^{p_2-p_1} \rho_F^{p_2-1} \tag{10}$$

Note that (8) implies the turbulent strength parameters are related as $U_2 = U_1 \mu_0^{p_2-p_1}$. Using (2) and (9) the intensity spectrum can be written in dimensionless form as

$$I(\mu) = 2\int_0^\infty \exp\{-\gamma(\eta,\mu)\} \cos(\eta\mu)\, d\eta \tag{11}$$

and the structure interaction term becomes

$$\gamma(\eta,\mu) = 16\int_0^{\mu_0} U_1 \chi^{-p_1} \sin^2(\chi\eta/2) \sin^2(\chi\mu/2) \frac{d\chi}{2\pi} + 16\int_{\mu_0}^{\infty} U_2 \chi^{-p_2} \sin^2(\chi\eta/2) \sin^2(\chi\mu/2) \frac{d\chi}{2\pi} \tag{12}$$

In (12) the variable of integration $\chi = q\rho_F$ is a normalized wavenumber (like μ) and the integration is partitioned at the spectral break, which occurs at the dimensionless wavenumber $\mu_0 = q_0\rho_F$.

The normalized phase and intensity spectra are related to their dimensioned counterparts as:

$$\begin{aligned} P(\mu) &= \Phi_{\delta\phi}(\mu/\rho_F)/\rho_F \\ I(\mu) &= \Phi_I(\mu/\rho_F)/\rho_F \end{aligned} \tag{13}$$

Once the intensity spectrum (11) has been evaluated, the dimensional form of the spectrum can be recovered using (13). Lastly, we define the scattering strength U^* as the normalized phase spectral power at the Fresnel scale

$$U^* = \begin{cases} U_1, & \mu_0 \ge 1 \\ U_2, & \mu_0 < 1 \end{cases} \tag{14}$$

In other words, $U^* \equiv P(\mu=1)$. For an unmodified power law, $U_1=U_2=U^*$ so we sometimes write just U (i.e. we omit the asterisk). When $U^*<1$ the scatter is weak and when $U^*>1$ the scatter is strong. The four parameters p_1, p_2, μ_0, and U^* specify all solutions for 2-component spectra, in that different combinations of perturbation strength, propagation distance, and frequency produce exactly the same results.

2. ASYMPTOTIC RESULTS

Given the two-component structure model in (9), we derived analytic expressions for the structure function $D_{\delta\phi}(\xi)$ and the structure interaction function $\gamma(\eta,\mu)$. Asymptotic techniques similar to those described in [Rino, 1979; Rino and Owen, 1984] were then applied to determine the asymptotic behavior of the S_4 index and intensity correlation length, ξ_c, as U^* grows large. This analysis is lengthy and omitted here for brevity (the details will be published elsewhere).

S_4 *index*

For the two component power law model we can show, using asymptotic methods, that for $1<p_1<3$ the limiting value of S_4 is unity (although a local maximum exceeding unity may be achieved for intermediate values of the scattering strength), whereas for $3<p_1<5$ it is

$$S_4 \to \sqrt{\frac{p_1-1}{5-p_1}}, \quad 3<p_1<5 \tag{15}$$

The quasi-saturation state whereby S_4 approaches a value exceeding unity in the strong scatter limit occurs only when p_1>3, and when it occurs the value of p_2 is immaterial (i.e. the large scale phase structure dominates the development via strong focusing). Rino and Owen [1984] derived a similar result for the case of an unmodified power law spectrum which is less than the limiting value in (15) because the low frequency contribution was neglected in their analysis. In fact, the low frequency contribution becomes increasingly significant as p exceeds 3 due to strong focusing and should be taken into account.

Correlation length

If p_1<3 then $S_4 \to 1$ and $\rho(\xi) = [R(\xi)-1]/S_4^2 \to \exp\{-D_{\delta\phi}(\xi)\}$ in the limit of asymptotically strong scatter (the proof will be presented elsewhere). Under these conditions the asymptotic intensity correlation length can be determined by solving (either analytically or numerically) the following for ξ_c:

$$\exp\{-D_{\delta\phi}(\xi_c)\} = 1/2 . \tag{16}$$

The limiting ξ_c exhibits a power law dependence on the scattering strength of the form $\xi_c \propto U^{*-n_c}$, where we define n_c as the "spectral index" of the limiting correlation length.

When p_1<p_2<3 the limiting correlation length takes the following form:

$$\xi_c \to \left[\frac{\log(2)}{U^* A_1} \left(\frac{\sqrt{\pi}(p_2-1)\Gamma\left[\frac{p_2}{2}\right]}{2^{2-p_2}\Gamma\left[\frac{3-p_2}{2}\right]} \right) \right]^{1/(p_2-1)} \tag{17}$$

where A_1 = 1 if μ_0≤1 and $A_1 = \mu_0^{p_2-p_1}$ if μ_0>1. In this case, the correlation length spectral index is n_c=1/(p_2-1) which tends to 1/2 as p_2 approaches 3 (the case p_2 = 3 is special, however). If p_1=p_2 we recover the result for an unmodified power spectrum. It is clear from (17) that limiting correlation length is dictated by the high frequency portion of the phase spectrum, as might be expected. In fact, if μ_0≤1 the limiting correlation length is the same as for an unmodified power law with p=p_2.

When p_1<3, p_2>3 the limiting correlation length is

$$\xi_c \to \left[\frac{\log(2)}{U^* A_2} \left(\frac{\pi(3-p_1)(p_2-3)}{(p_2-p_1)} \right) \right]^{1/2} \tag{18}$$

where $A_2 = \mu_0^{3-p_2}$ if μ_0≤1 and $A_2 = \mu_0^{3-p_1}$ if μ_0>1. In this case the correlation length spectral index is n_c =1/2 regardless of the values of p_1 and p_2 (so long as p_1<3, and p_2>3). This dependence on U* is due to the leading term in the Taylor series expansion of the structure function, which depends quadratically on the spatial separation under these conditions. We note that the rate of decrease in ξ_c with increasing scattering strength is slower when a spectral break is present than for an unmodified power law with *any* slope. This is true whether the break acts as an outer scale or an inner scale. In nature we expect to encounter the behavior described by (18) rather than (17) since all real turbulent plasmas have outer and inner scales.

If p_1≥3 the structure function does not exist, and alternative methods are required to obtain the limiting form of the correlation function. The result of this analysis (not shown here) is that the correlation length spectral index is n_c=1/(5-p_1), which depends only on the low-frequency phase structure.

3. NUMERICAL RESULTS

Since the integral appearing in (11) is oscillatory over the semi-infinite domain, specialized quadrature methods are required for its efficient numerical evaluation. Having experimented with many techniques, we found the most effective to be the double exponential quadrature scheme developed by Ooura and Mori [1991]. To prevent precision loss when evaluating the integrand in (11) when the scatter is strong, and because round-off errors become increasingly important to control for small and large frequencies, we used the mpmath Python library for arbitrary-precision floating-point arithmetic to perform the numerical quadrature. Even still we found it necessary to expand the structure interaction function in a Taylor series (summing terms until convergence was reached), when evaluating this function for large arguments, in order to minimize round-off errors due to the subtraction of nearly equal terms.

Intensity Spectrum for an Unmodified Power Law

Figure 1 shows the computed intensity spectra for a few different scattering strengths. Results are compared for phase spectra following an unmodified power law with shallow (p=2.5) and steep (p=4) slope. Note that significant departures from power law behavior develop as the scattering becomes strong ($U \gg 1$). In particular, the shallow slope case develops a prominent low frequency enhancement, while the steeply sloped case develops significant spectral broadening at high frequencies. This spectral broadening can result in quasi-saturation states with S_4 exceeding 1. For asymptotically small and large μ the intensity spectrum behaves like $I(\mu) \sim \mu^{4-p}$ and $I(\mu) \sim 2\mu^{-p}$, respectively. These asymptotes are shown as gray dashed lines in Figure 1. The return to power law behavior at large frequencies is gradual for shallow spectra (p<3) and abrupt for steep spectra (p>3).

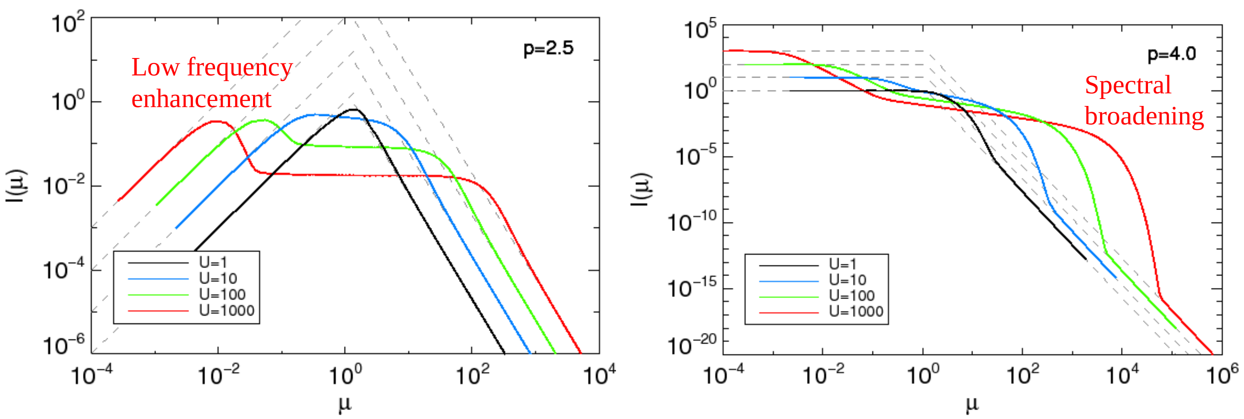

Figure 1. Normalized intensity spectra for an unmodified power law with spectral index 2.5 (left) and 4.0 (right). The different curves correspond to the scattering strength values U=1, 10, 100, and 1000.

Effect of an Outer or Inner Scale

Next we repeated the simulations shown in Figure 1, for the case with U^*=1000, but this time we imposed a spectral break at μ_0=0.1 to illustrate the effect an outer scale. The results for several values of p_1 are shown in Figure 2. When a spectral break is present, the intensity spectrum for asymptotically small and large μ behaves like $I(\mu) \sim \mu^{4-p_1}$ and $I(\mu) \sim 2\mu^{-p_2}$, respectively. The spectral break increasingly erodes large scale irregularity structure as p_1 is decreased. Eroding large scale structure suppresses development of the low frequency enhancement for the shallow spectrum case. For the steep spectrum case, it suppresses both low frequency fluctuations and also spectral broadening at high frequencies. From this we infer that strong focusing by large scale irregularity structure is responsible for this spectral broadening via the generation of small scale fluctuations in the intensity field.

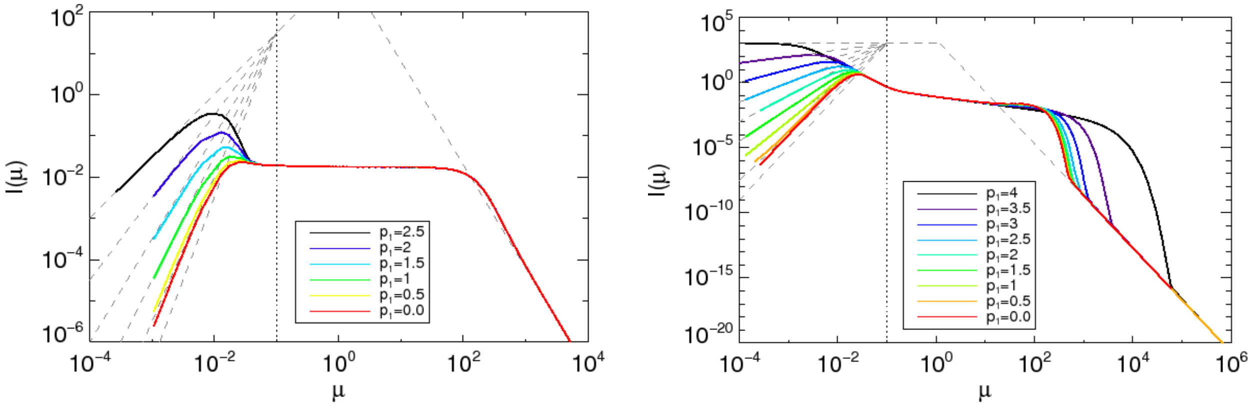

Figure 2. Intensity spectrum as a function of low frequency spectral index p_1 when the high frequency spectral index is p_2=2.5 (left) and p_2=4.0 (right). The scattering strength is U^*=1000. Dashed lines show theoretical low and high frequency asymptotes. The vertical dotted line indicates the break wavenumber.

Again we repeated the simulations shown in Figure 1, for the case with U^*=1000, but this time we imposed a spectral break at μ_0=10 to illustrate the effect of an inner scale. The results for several values of p_2 are shown in Figure 3. The spectral break increasingly erodes small scale irregularity structure as p_1 is decreased. For shallow spectra (p<3) this reduces the width of the intensity spectrum and increases the correlation length compared to that for an unmodified power law spectrum. Furthermore, it promotes focusing by the larger scale structure that remains. Note that this focusing increases the fluctuation power in the intensity field at all scales between the low frequency return to power law and the spectral break. A number of researchers (Bhattacharyya, et al., 1992) have observed that the presence of small scale structure associated with fully developed plasma turbulence at low-latitudes in the early post-sunset hours tends to suppress strong focusing and inhibit S_4 values exceeding unity. Later in the evening, diffusive processes erode the small scale structure, which encourages strong focusing and can result in S_4 values well above unity. Figure 3 (right) shows that for steep spectra (p>3) the addition of an inner scale has essentially no effect, except at frequencies exceeding the spectral break.

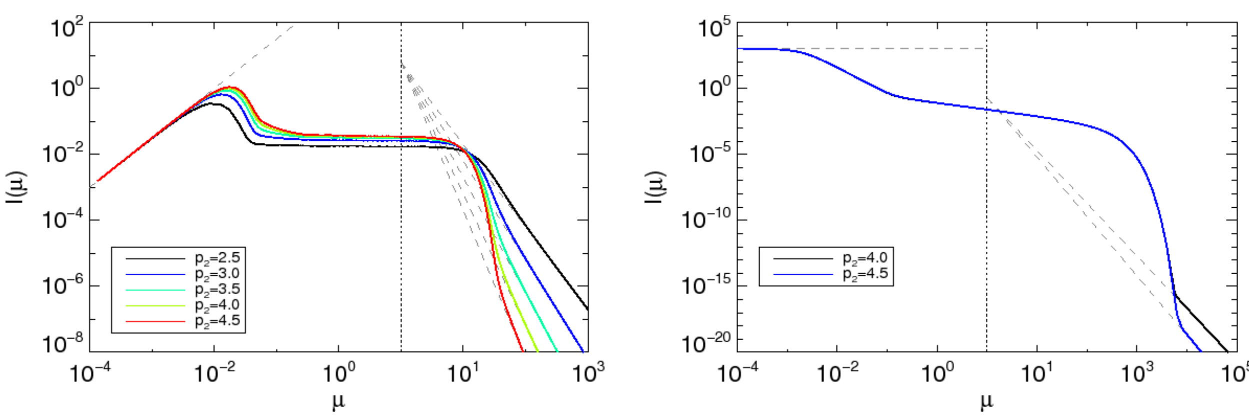

Figure 3. Intensity spectrum as a function of high frequency spectral index p_2 when the high frequency spectral index is p_1=2.5 (left) and p_1=4.0 (right). The scattering strength is U^*=1000. Dashed lines show theoretical low and high frequency asymptotes. The vertical dotted line indicates the break wavenumber.

S_4 as a Function of Scattering Strength

Figure 4 shows the dependence of S_4 on the scattering strength for an unmodified power law with different values of the spectral index *p*. When $p<3$ the S_4 index saturates at unity (with overshoot if $p>2$). Power law spectra with $p>3$ admit sustained quasi-saturation states with $S_4>1$. The circles indicate the theoretical limiting values given in (15). The agreement with theory is excellent, except for the case $p=3$. The reason for the apparent discrepancy is that when $p=3$ the S_4 decreases to unity from above extremely slowly (we confirmed this by increasing the scattering strength in increasing steps to $U^{*}=10^{12}$).

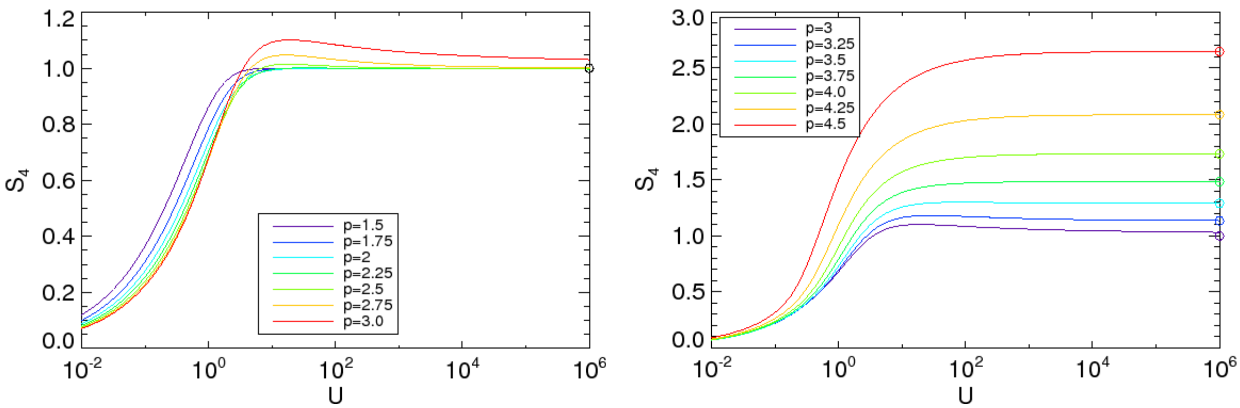

Figure 4. S_4 versus turbulence strength *U* for an unmodified power law with spectral index varying from 1.5 to 3.0 (left) and from 3.0 to 4.5 (right). Open circles shown along the right axis indicate theoretical limiting S_4 values.

Next, we repeated the simulations shown in Figure 4 but with an outer scale ($p_1=0$) introduced at $\mu_0=0.1$. The results are shown in Figure 5. For shallow spectra ($p<3$), the outer scale has only the minor effect of causing the saturation state $S_4=1$ to be approached more quickly as the scattering strength increases. For steep spectra ($p>3$) the effect of the outer scale is more dramatic. The suppression of large scale irregularity structure mitigates the quasi-saturation state and causes S_4 to saturate at unity. From this we infer that strong focusing by large scale structure is responsible for quasi-saturation states with $S_4>1$.

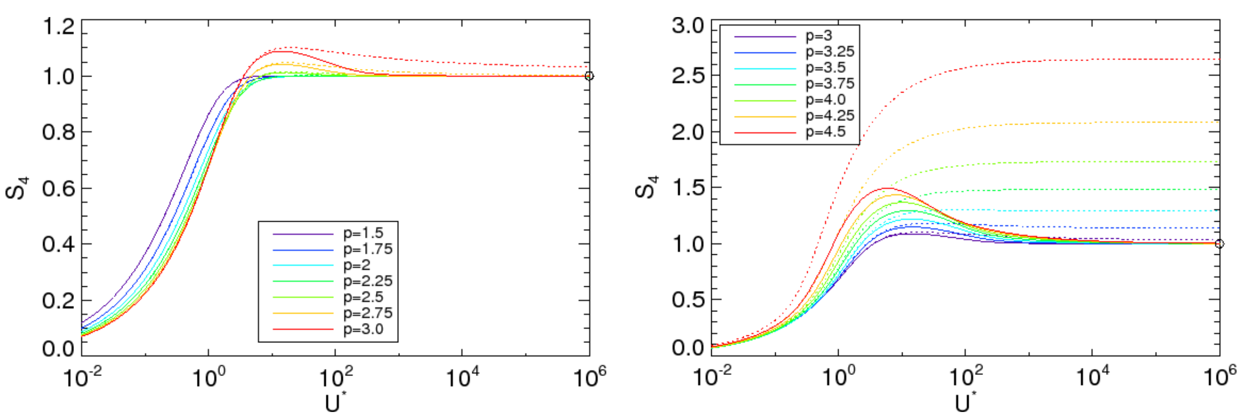

Figure 5. S_4 versus turbulence strength *U* for a modified power law with outer scale wavenumber $\mu_0=0.1$ and spectral index varying from 1.5 to 3.0 (left) and from 3.0 to 4.5 (right). Open circles shown along the right axis indicate theoretical limiting S_4 values. The dashed curves show the S_4 for the corresponding unmodified power law (e.g. Figure 4) for comparison.

Correlation Length as a Function of Scattering Strength

Figure 6 shows the dependence of the normalized intensity correlation length on the scattering strength for an unmodified power law. The dashed lines show the theoretical limiting correlation length. For the cases with $p<3$, the theoretical correlation length is that gievn in (17); for the cases with $p\geq3$ it is for the result (not given here) that holds when the structure function does not exist. In both cases, the agreement with the theoretical results is excellent when U^* is large.

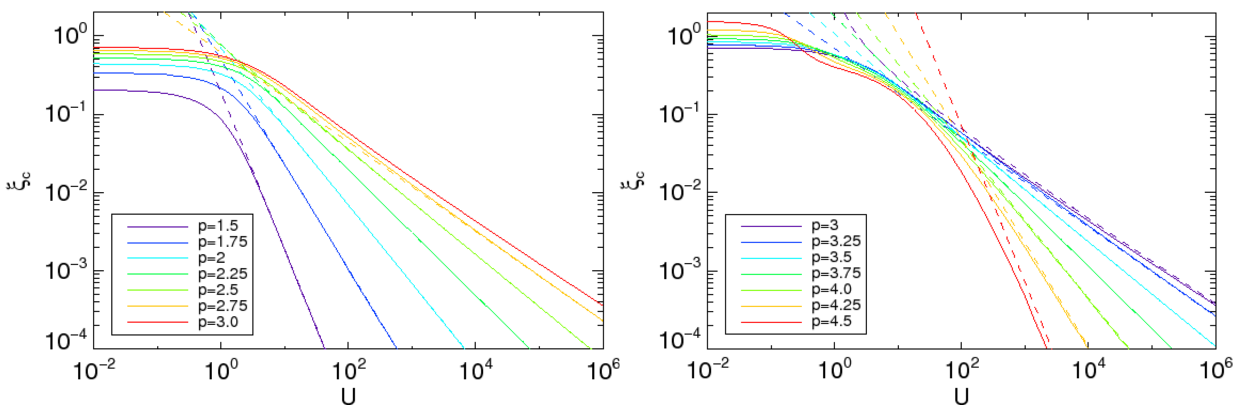

Figure 6. Normalized correlation length versus turbulence strength U for an unmodified power law with spectral index varying from 1.5 to 3.0 (left) and from 3.0 to 4.5 (right).

Figure 7 shows the dependence of the normalized intensity correlation length on the scattering strength when an outer scale is present. For shallow spectra ($p<3$), the correlation length is practically unaffected by presence of the outer scale. For steep spectra ($p>3$), the outer scale mitigates the production of small scale intensity fluctuations via strong focusing which significantly increases the correlation length (compared to the case of an unmodified power law). The theoretical result (18) is in excellent agreement with the simulation results for large U^*. Although the results are not shown here, when we impose an inner scale, rather than an outer scale, we find the correlation length is affected for shallow spectra ($p<3$) but not for steep spectra ($p>3$). For the case $p=3$ the correlation length is relatively insensitive to either an outer scale or inner scale.

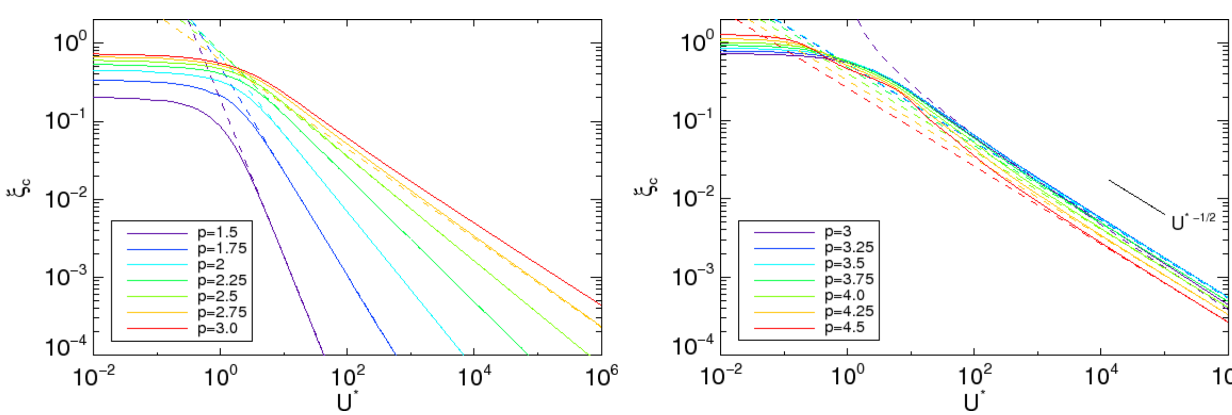

Figure 7. S_4 versus turbulence strength U^* for a modified power law with outer scale wavenumber μ_0=0.1 and spectral index varying from 1.5 to 3.0 (left) and from 3.0 to 4.5 (right).

Figure 8 shows the variation of the limiting correlation length spectral index as a function of phase spectral index for the case of an unmodified power law (left) and when a spectral break is present with p_1<3, p_2>3 (right). The spectral break in the latter case (right) may represent either a steep spectrum with an outer scale or a shallow spectrum with an inner scale. In this case, we measured the slopes of the curves shown in Figures 6 and 7. These simulation results are compared with the theoretical results for an unmodified power law which predict that $n_c=1/(p-1)$ when $p<3$ and $n_c=1/(5-p)$ when $p>3$. The variation of n_c as a function of p is symmetric about the result for $p=3$. When a spectral break with $p_1<3$, $p_2>3$ is present this symmetry is broken. In this case the intensity correlation length decreases with increasing scattering strength such that $n_c=1/2$. As can be seen in the figure, the agreement between the simulations (dots) and the theoretical results (dashed curves) is excellent.

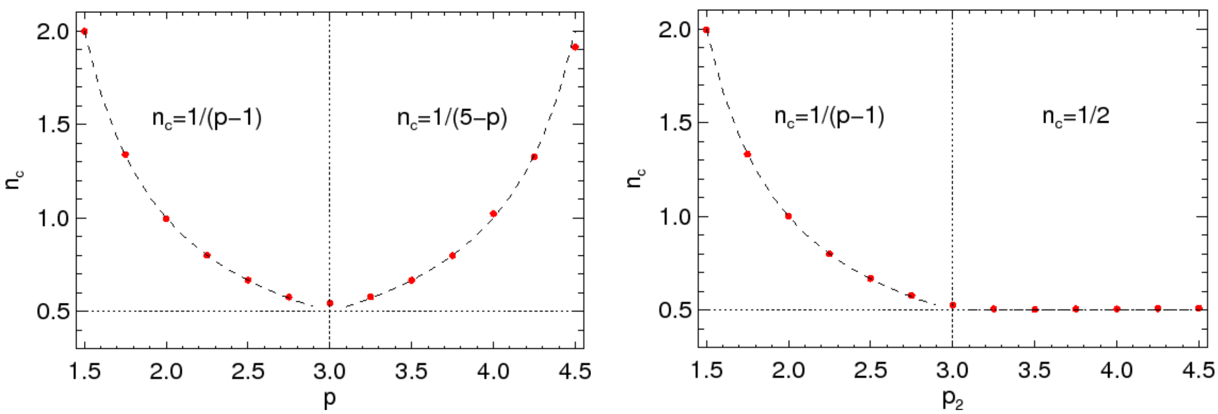

Figure 8. Spectral index of the power law (n_c) representing the limiting intensity correlation length. The left plot is for an unmodified power law, the right plot for the case when a spectral break with p_1<3, p_2>3 is present. The dots show simulation results, whereas the dashed lines are theoretical curves.

General Two-Component Spectra

Next we investigated the effect of an arbitrarily located spectral break (i.e. not necessarily restricted to an outer scale or inner scale) on the development of S_4 and ξ_c as a function of scattering strength. Figure 9 shows the results for the case $p_1=2$, $p_2=4$ with μ_0 ranging logarithmically from 0.01 to 100. The curves corresponding to the results for unmodified power spectra with $p=2$ and $p=4$ are also shown for comparison. Since a spectral break is present with $p_1<3$ we observe that S_4 saturates at unity, in accord with theory. The theoretical correlation length predicted by (17) is shown in Figure 9 using dashed lines, and they follow the simulation results closely for large U^*. For a general two-component spectrum, the location of the spectral break relative to the Fresnel scale dictates the development of the intensity field. The simulations with $\mu_0<1$ behave like a power law with $p=4$ and an outer scale, whereas the simulations with $\mu_0>1$ behave as a power law with $p=2$ and an inner scale. We observe that an inner scale has a minimal effect on S_4 when $p<3$, whereas an outer scale has a dramatic effect on S_4 when $p>3$ (in that the quasi-saturation state is mitigated). We also observe that introducing an inner scale to a power law with $p=p_1<3$ increases the correlation length when the scatter is strong. Similarly, introducing an outer scale to a power law with $p=p_2>3$ increases the correlation length when the scatter is strong.

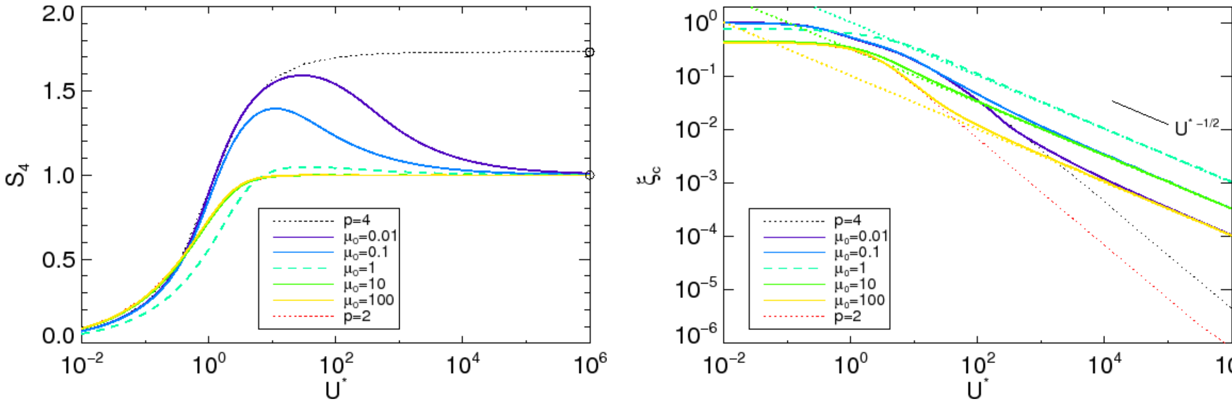

Figure 9. S_4 (left) and correlation length (right) for a modified spectrum with p_1=2, p_2=4 and break wavenumber μ_0 ranging from 0.01 to 100. Also shown for comparison are results for unmodified power law spectra with *p*=2 and *p*=4. Open circles along the right axis indicate theoretical limiting S_4 values.

Finally, we investigated the effect of the break wavenumber for the case p_1=3.5, p_2=4, with μ_0 ranging logarithmically from 0.01 to 100. Note that the limiting S_4 in Figure 10 is not unity for the two-component case, but instead the value predicted by (15) which is $(5/3)^{1/2}$. Furthermore, note for the cases with μ_0>1 that the correlation length follows the curve corresponding to an unmodified power law with *p*=3.5, in accord with the theory. The slight change in slope between p_1 and p_2 is not sufficient to mitigate the development of small scale structure via strong focusing in this case.

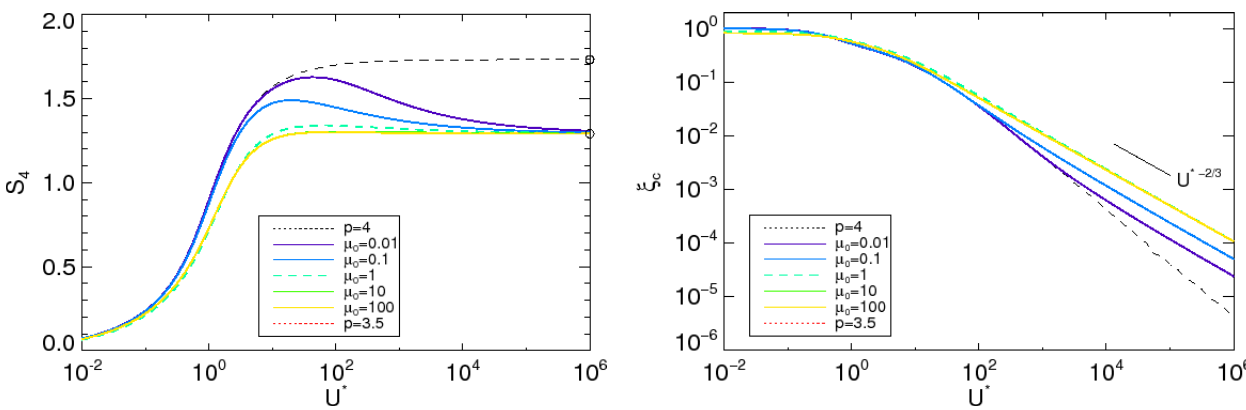

Figure 10. S_4 (left) and correlation length (right) for a modified spectrum with p_1=3.5, p_2=4 and break wavenumber μ_0 ranging from 0.01 to 100. Also shown for comparison are results for unmodified power law spectra with *p*=3.5 and *p*=4. Open circles along the right axis indicate theoretical limiting S_4 values.

4. CONCLUSIONS

We have extended the phase screen power law theory for ionospheric scintillation to account for the case where the refractive index irregularities follow a two-component power law spectrum. Using this spectral model, we solved the 4^{th} moment equation governing the intensity fluctuations for the case of two-dimensional field-aligned ionospheric irregularities. A specific normalization was invoked to exploit the self-similar properties of the problem to achieve a universal scaling, such that different combinations of perturbation strength, propagation distance, and frequency produce exactly the same results. The numerical algorithm was used to validate new theoretical predictions for the behavior of the scintillation index and intensity correlation length under strong scatter conditions.

We make the following observations which are valid in the asymptotic strong scatter limit. For steeply sloped spectra with $p_2>p_1>3$ the scintillation index approaches a quasi-saturation state exceeding unity, $S_4\rightarrow((p_1-1)/(5-p_1))^{1/2}$, as the strength of scatter increases. The presence of a spectral break with $p_1<3$ causes the S_4 index to recede from its maximum with increasing scattering strength to ultimately saturate at unity. When a spectral break is present the rate of decrease in ξ_c with increasing scattering strength is slower than for an unmodified power law with any slope. We expect this slower rate of decrease to be observed in nature, since all real turbulent plasmas have outer and inner scales.

In general, when $p_1<p_2<3$ the limiting behavior of the intensity statistics is dictated by the high frequency portion of the irregularity spectrum (i.e. the influence of scales sizes larger than the Fresnel scale becomes insignificant), whereas when $p_2>p_1>3$ it is dictated by the low frequency portion of the irregularity spectrum (i.e. the influence of scales sizes smaller than the Fresnel scale becomes insignificant). As a consequence, shallow power-law spectra ($p<3$) are insensitive to an outer scale but are sensitive to an inner scale. Steep power-law spectra ($p>3$) are insensitive to an inner scale but are sensitive to an outer scale. Power law spectra with $p\approx3$ are relatively insensitive to both outer and inner scales. For the general case with $p_1<3$, $p_2>3$, irregularity scales sizes both smaller and larger than the Fresnel scale can contribute to the intensity statistics. In this case, the location of the spectral break relative to the Fresnel scale dictates the development of the intensity field.

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