Assimilative Model for Ionospheric Dynamics Employing Delay, Doppler, and Direction of Arrival Measurements from Multiple HF Channels

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The Ionospheric Reconstruction Problem: Tikhonov Method

\[ N(\mathbf{r}, t) = N_0(\mathbf{r}, t)Q(u(\mathbf{r}, t)); \quad \text{i.e. } Q(u(\mathbf{r}, t)) = e^{u(\mathbf{r}, t)} \]
\[ U = \{u(\mathbf{r}, t)\} \]
\[ Y \approx M[U] \]

\( Y \) is the set of measured values obtained via various ionospheric measurements (such as TEC data, HF oblique propagation delay).

The solution must fit the data within errors of measurements.

\[(Y - M[U])^T S^{-1} (Y - M[U]) / \text{dim}(Y) \leq 1\]

The pseudo-covariance \( P \) matrix is defined in such a way that the stabilizing functional tends to take on larger values for unreasonably behaving solutions (“reasonable” \( \Leftrightarrow \) “smooth”).

The nonlinear optimization problem is solved iteratively (Newton-Kontorovich).

There are infinitely many such solutions:

The smoothest solution is selected by minimizing the stabilizing functional

\[ \mathbf{U}^T P^{-1} \mathbf{U} \rightarrow \min \]
Simulated values of measured data can be obtained for any ionospheric model $U$ via numerical ray tracing (RT).

This defines the non-linear functional of measurements $M[U]$

Ray Tracing Equations

Hamiltonian Formulation of RT Equations [Haselgrove, 1957, Jones, 1975]

$$\frac{d\mathbf{R}}{d\tau} = -\left(\frac{\partial H}{\partial k}\right) / \left(\frac{\partial H}{\partial \omega}\right) ,$$

$$\frac{d\mathbf{k}}{d\tau} = \left(\frac{\partial H}{\partial \mathbf{R}}\right) / \left(\frac{\partial H}{\partial \omega}\right) ,$$

$$\frac{dg}{d\tau} = c,$$

$$\frac{d\omega}{d\tau} = -\frac{\partial H}{\partial t} / \left(\frac{\partial H}{\partial \omega}\right)$$

- Group path equation

- Doppler equation

$$d\mathbf{X} = F(\mathbf{X}, [N, \frac{\partial N}{\partial t}]) \quad \mathbf{X} = [X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8]^T$$

wave vector

Doppler

position

group path
- The non-linear inverse problem is solved iteratively as a sequence of linear problems. At the iteration $n$ the non-linear functional $M[U]$ is approximated by a linear operator $L$ as follows

$$M[U] = M[U_{n-1}] + L(U - U_{n-1}) + o\left(\|U - U_{n-1}\|\right) \quad \Leftrightarrow \quad L = \delta M / \delta U$$

- $L$ is the Ray Path Response (RPR) operator

- The Ray Path Response operator $L$ is estimated using the extended RT equations – the equations augmented with the linearized ray-tracing equations

**Extended RT Equations**

$$\frac{dX}{dt} = F(X, [N, \frac{\partial N}{\partial t}])$$

$$\frac{dA}{d\tau} = B(X, [N, \frac{\partial N}{\partial t}])A$$

$$B_{ij} = \frac{\partial F_i}{\partial X_j}\bigg|_{x=x(\tau)}$$

$$A_{ij}\bigg|_{\tau=0} = \delta_{ij}, \quad i, j \in [1:8]$$

8+8x8=82 equations in the extended system
Evaluation of the Ray Path Response Operator $L$

The 3-D ionospheric model is specified by $U_{n-1}$

Perform ray-tracing (ray-homing) for the Rx-Tx configuration. Determine ray exit direction

Integrate the extended RT equations at the ray exit direction found at ray homing

Evaluate linear response $K(\tau')$ of parameters measured by the RX

Produce the Green function $G(\tau, \tau')$ for the boundary-value problem (subject to the constraint that the ray end points are fixed in space)

Produce impulse response function $G^I(\tau, \tau')$ of the linearized RT equations

Evaluate RPR

$$L^S = \int d\tau' K(\tau') \left( \frac{\partial F(X, [N, \dot{N}])}{\partial N} N_0 Q'(u_{n-1}) + \frac{\partial F(X, [N, \dot{N}])}{\partial \dot{N}} \left( N_0 Q''(u_{n-1}) \dot{u}_{n-1} + \dot{N}_0 Q'(u_{n-1}) \right) \right)$$

$$L^D = \int d\tau' K(\tau') \frac{\partial F(X, [N, \dot{N}])}{\partial \dot{N}} N_0 Q'(u_{n-1})$$

Where $N = N_0(r, t)Q(u_{n-1}(r, t))$
Evaluation of RPR within the 3-D ionospheric inversion problem can be performed with little computational burden because it is reduced to computation of several one-dimensional integrals.

For HF data the main computational burden remains associated with the classical ray homing task. This task precedes computation of RPR.

Numerical representation of the RPR is a sparse matrix with non-zero elements occupying only nodes of the spatial grid that are adjacent to the ray trajectories that connect receiver and transmitter. Matrix operations with RPR are not a substantial computational burden as we take advantage of the sparse character of RPR.
Test with Range-Doppler Data Set

- **Florida Collection (August 13, 2013)**
  - Range/Doppler Data
  - Receiver at Vero Beach, FL
  - Multiple transmitter sites
  - Three hour collection of 3 KHz
  - bandwidth FMCW waveform

![Map of Florida with locations marked](image)
Range-Doppler Data of August 13, 2013
Employed by GPSII
Range-Doppler Data Compared to GPSII Fit
GPSII Solution with Range-Doppler Data

Plasma Frequency (MHz) at 220 km; 13 Aug 2013, 13:36 UT
Assimilation of Oblique Ionograms (along with TEC and VI data)

Geography of DSTO data sources employed in this test*

*Only a subset of OI links maintained by DSTO was utilized
Comparison of GPSII results for 2 links

GPSII runs with oblique ionogram reproduce observed OI details (yellow) for both assimilated (left) and test (right) propagation links.

Both links shown at 01:45. All synthetic OIs are extraordinary-ray traces.

Scherger-Alice Springs (North-south; data link)
Lynd River-Wyndham (East-west; test link)
Impact of OI data on GPSII Model

Vertical cut through the model at latitude -18 degrees

Inversion with OI Data
Conclusions

• The theoretical framework for incorporating HF channel probe data (propagation delay, angles of arrival, Doppler shift) into ionospheric inversion algorithms has been developed

• Capabilities to assimilate data from HF channel probes and oblique ionograms have been added to GPSII

• Performance and validation of the algorithm are addressed in the companion paper by L.J. Nickish
Backup
Evaluation of the Ray Path Response Operator $L$

- The 3-D ionospheric model is specified by $U_{n-1}$.
- Perform ray-tracing (ray-homing) for the Rx-Tx configuration. Determine ray exit direction.
- Integrate the extended RT equations at the ray exit direction found at ray homing.
- Produce impulse response function $G^{I}(\tau, \tau')$ of the linearized RT equations.
- Produce the Green function $G(\tau, \tau')$ for the boundary-value problem (subject to the constraint that the ray end points are fixed in space).
- Evaluate linear response $K(\tau')$ of parameters measured by the RX.

**DETAILS**

$G^{I}(\tau, \tau') = \begin{cases} 
0, \tau < \tau' \\
A(\tau)A^{-1}(\tau'), \tau \geq \tau'
\end{cases}$

$\tau_L = \text{propagation delay at the landing (Rx) point}$

$K(\tau') = G_{4;8,1;8}(\tau_L, \tau') + F_{4;8}(\tau_L)D_{4;1;8}(\tau')$

Assuming that RX measures direction of arrival, group path, Doppler.
GPSII Solution with Range-Doppler Data

Plasma freq. (MHz) at Latitude 28.00; 13 AUG 2013, 13:36 UT
GPSII Solution with Range-Doppler Data
GPSII Solution with Range-Doppler Data and the GPS TEC data
Reconstruction of a TID from Simulated OTHR Data

Plasma Frequency (MHz) at 250 km; 15 Jan 2014, 19:00 UT

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\[
\delta U \bigg|_t = \delta U(t_m) \frac{t_m - t}{t_m - t_{m-1}} + \delta U(t_{m-1}) \frac{t-t_{m-1}}{t_m - t_{m-1}}; \quad \dot{\delta U} \bigg|_t = \delta U(t_m) \frac{1}{t_m - t_{m-1}} - \delta U(t_{m-1}) \frac{1}{t_m - t_{m-1}}
\]